

## On the Primal and Dual Constraint Sets in Geometric Programming

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Geometric programming, as developed by Duffin, Peterson, and Zener [1], is a nonlinear programming theory with broad applications. See, e.g., [1-4]. An important aspect of geometric programming is the existence of strong duality theorems similar to those of linear programming. In this paper, we show that some recent results concerning the primal and dual feasible and optimal sets in linear programming [5, 6] also have their counterpart in geometric programming. This leads to characterization of compactness of the feasible or optimal sets upon which further existence theorems might be based.

Consider the following pair of primal and dual geometric programs:

### *Primal Program*

(P) Minimize

$$g_0(x) = \sum_{j \in J(0)} p_{j0}(x) \quad (1a)$$

subject to

$$g_k(x) = \sum_{j \in J(k)} p_{jk}(x) \leq 1, \quad k = 1, \dots, K \quad (1b)$$

$$x = (x_1, \dots, x_m) > 0 \quad (1c)$$

where

$$p_{jk} = c_{jk} \prod_{i=1}^m x_i^{a_{ijk}}, \quad c_{jk} > 0, \quad j \in J(k), \quad k = 0, \dots, K. \quad (1d)$$

### *Dual Program*

(D) Maximize

$$v(\delta, \lambda) = \left\{ \prod_{k=0}^K \prod_{j \in J(k)} \left( \frac{c_{jk}}{\delta_{jk}} \right)^{\delta_{jk}} \right\} \prod_{k=1}^K \lambda_k^{\lambda_k} \quad \left( \text{where } \lambda_k = \sum_{j \in J(k)} \delta_{jk} \right) \quad (2a)$$

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subject to

$$\sum_{J(0)} \delta_{j0} = 1 \quad (2b)$$

$$\sum_{k=0}^K \sum_{J(k)} a_{ijk} \delta_{jk} = 0, \quad i = 1, \dots, m \quad (2c)$$

$$\delta_{jk} \geq 0, \quad j \in J(k), \quad k = 0, \dots, K. \quad (2d)$$

The set of vectors  $x$  satisfying (1b) and (1c) is called the *primal feasible set*. Similarly, the set of vectors  $\delta$  satisfying (2b), (2c), and (2d) is called the *dual feasible set*. Either program is said to be *consistent* if its feasible set is nonempty. If (P) is consistent, the constrained infimum of  $g_0(x)$  is called the *infimum of (P)*. The following two lemmas will be useful in the subsequent analysis.

LEMMA 1. *Let the primal program (P) be consistent. Then it has a positive infimum if and only if (D) is consistent.*

*Proof.* Follows directly from Theorems 6.1.1, 6.1.2 and Lemma 6.1.1 [1, pp. 166–7]. Q.E.D.

LEMMA 2. *The variable  $z_j$  is unbounded in the set of solutions to the consistent constraints,*

$$Gz = b, \quad z \geq 0 \quad (3)$$

where  $G$  is a finite matrix and  $b$  is a vector, if and only if there exists a solution  $\rho$  to

$$G\rho = 0, \quad \rho \geq 0 \quad (4)$$

with  $\rho_j > 0$ .

*Proof.* For any solution  $\bar{z}$ ,  $z = (\bar{z} + \theta\rho)$  is a solution for all nonnegative real  $\theta$ . If  $\rho_j > 0$ , then  $z_j \rightarrow \infty$ . On the other hand, all solutions to (3) can be written

$$z = \sum_{\nu} z^{\nu} \lambda_{\nu} + \sum_t \rho^t \mu_t$$

where the  $z^{\nu}$  are the basic solutions, and the  $\rho^t$  are the extreme rays, i.e., the  $\rho^t$  satisfy (4). Moreover, the ranges of  $\nu$  and  $t$  are finite [7]. Therefore, for some  $t$ ,  $\rho_j^t > 0$ ; otherwise  $\max_{\nu} z_j^{\nu}$  would be a bound for  $z_j$ . Q.E.D.

Our first main result is

THEOREM 1. *Let the primal geometric program (P) be consistent and have a positive infimum. Then the term  $p_{rs}(x)$ , for  $s \geq 1$ , is bounded away from zero in the primal feasible set if and only if  $\delta_{rs}$  is unbounded in the dual feasible set.*

*Proof.* The term  $p_{rs}(x)$  is bounded away from zero in the primal feasible set if and only if the geometric program

$$\text{Minimize } p_{rs}(x) \quad (5a)$$

subject to  $x > 0$ ;

$$\sum_{j \in (k)} p_{jk}(x) \leq 1, \quad k = 1, \dots, K \quad (5b)$$

has a positive infimum, and by Lemma 1 this is the case if and only if the dual program is consistent, i.e., there exists a solution to

$$\delta_0 = 1 \quad (6a)$$

$$a_{irs}\delta_0 + \sum_{k=1}^K \sum_{j \in (k)} a_{ijk}\delta_{jk} = 0, \quad i = 1, \dots, m \quad (6b)$$

$$\delta_{jk} \geq 0, \quad j \in J(k), \quad k = 1, \dots, K. \quad (6c)$$

On the other hand, by Lemma 2,  $\delta_{rs}$  is unbounded in the dual feasible set (2b)–(2d) if and only if there exists a solution, with  $\rho_{rs} > 0$ , to the homogeneous equations

$$\sum_{j \in (0)} \rho_{j0} = 0 \quad (7a)$$

$$\sum_{k=1}^K \sum_{j \in (k)} a_{ijk}\rho_{jk} = 0, \quad i = 1, \dots, m \quad (7b)$$

$$\rho_{jk} \geq 0, \quad j \in J(k), \quad k = 0, \dots, K. \quad (7c)$$

A solution to (6) may be constructed from a solution to (7) by defining  $\delta_{jk} = \rho_{jk}/\rho_{rs}$  for  $(j, k) \neq (r, s)$ , and  $\delta_{rs} = 0$ ,  $\delta_0 = 1$ . A solution to (7) may be constructed from a solution to (6) by taking  $\rho_{j0} = 0$ ,  $\rho_{jk} = \delta_{jk}$ ,  $\rho_{rs} = 1 + \delta_{rs}$ .  
Q.E.D.

The second main result of this paper is

**THEOREM 2.** *Let the primal geometric program  $P$  be consistent and have a positive minimum. Then the term  $p_{rs}(x)$  is bounded away from zero in the primal optimal set if and only if  $\delta_{rs}$  is positive for some dual feasible solution.*

*Proof.* The term  $p_{rs}(x)$  is bounded away from zero in the optimal set if and only if the geometric program

$$\text{Minimize } p_{rs}(x) \quad (8a)$$

subject to

$$\frac{1}{M} \sum_{j \in (0)} p_{j0}(x) \leq 1 \quad (8b)$$

$$\sum_{j \in (k)} p_{jk}(x) \leq 1, \quad k = 1, \dots, K \quad (8c)$$

has a positive infimum. Here  $M$  is the minimum for program (P), so that (8b), (8c) describe the optimal set for program (P). By Lemma 1, the program (8) will have a positive infimum if and only if the dual program is consistent, i.e., if there exists a solution to

$$\delta_0 = 1 \quad (9a)$$

$$a_{irs}\delta_0 + \sum_{k=0}^K \sum_{J(k)} a_{ijk}\delta_{jk} = 0, \quad i = 1, \dots, m \quad (9b)$$

$$\delta_{jk} \geq 0, \quad j \in J(k), \quad k = 0, \dots, K. \quad (9c)$$

From a solution  $\delta$  to (9) a solution  $\delta$  to (2) with  $\delta_{rs} > 0$  may be constructed by first taking  $\delta_{jk} = \delta_{jk}$  for  $(j, k) \neq (r, s)$  and  $\delta_{rs} = 1 + \delta_{rs}$ . Then normalize, i.e., define  $\delta_{jk} = \delta_{jk} / (\sum_{J(k)} \delta_{jk})$ . On the other hand a solution  $\delta$  to (9) may be constructed from a solution  $\delta$  to (2) with  $\delta_{rs} > 0$ , by defining  $\delta_{jk} = \delta_{jk} / \delta_{rs}$  for  $(j, k) \neq (r, s)$ ,  $\delta_{rs} = 0$ ,  $\delta_0 = 1$ . Q.E.D.

From these theorems we obtain immediately:

**COROLLARY 1.** *Let the primal geometric program (P) be consistent and have a positive infimum. Then the primal feasible set is compact only if each of the  $\delta_{jk}$  for  $k \geq 1$  is unbounded on the dual feasible set.*

*Proof.* The primal feasible set is compact only if each  $p_{jk}(x)$  is bounded away from zero, i.e., if each  $\delta_{jk}$  is unbounded. Q.E.D.

**COROLLARY 2.** *Let the primal geometric program (P) be consistent. Then at most one of the feasible sets (primal or dual) is compact.*

*Proof.* If the primal feasible set is compact, the objective function must have a positive minimal value, and Corollary 1 applies. Q.E.D.

**COROLLARY 3.** *Let the primal geometric program (P) be consistent and have a positive minimum. Then the optimal solution set is compact only if there is a positive solution (i.e.,  $\delta_{jk} > 0$  all  $j, k$ ) to the dual constraints.*

*Proof.* Follows directly from Theorem 2. Q.E.D.

We note, finally, that by the last corollary, compactness of the primal optimal set therefore implies the "canonical" property in the sense of Duffin, Peterson, and Zener [1, p. 169].

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